Generalization of inequalities of Hardy and Littlewood

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Dedicated to Professor L. Rédei on his 70th birthday

The object of this paper is to give a generalization of some well-known and useful inequalities of HARDY and LITTLEWOOD [1].

We prove the following

Theorem. *Let $a_n \geqq 0$ and $\lambda_n \geqq 0$ $(n=1,2,\ldots)$ be given. Let $\nu_1 < \ldots < \nu_n < \ldots$ denote the indices for which $\lambda_{\nu_n} > 0$. Let N denote the number of the positive terms of the sequence λ_n, provided this number is finite; in the contrary case set $N=\infty$. Set $\nu_0 = 0$, and if $N < \infty$ then $\nu_{N+1} = \infty$. Using the notations*

$$A_{m,n} = \sum_{i=m}^{n} a_i \quad \text{and} \quad \Lambda_{m,n} = \sum_{i=m}^{n} \lambda_i \quad (1 \leqq m \leqq n \leqq \infty),$$

we have the following inequalities:

$$\left.\begin{aligned} &(1) \qquad \sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \leqq p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p \\ &(2) \qquad \sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \leqq p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p \end{aligned}\right\} \quad \text{for} \quad p \geqq 1$$

(the constant p^p being the best possible one) and

$$\left.\begin{aligned} &(3) \qquad \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p \leqq 8 \sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \\ &(4) \qquad \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p \leqq 9 \sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \end{aligned}\right\} \quad \text{for} \quad 0 < p \leqq 1.$$

If all the λ_n are positive then these inequalities reduce respectively to

$$\left.\begin{aligned} &(1') \qquad \sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \leqq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_{n,\infty}^p a_n^p \\ &(2') \qquad \sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \leqq p^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_{1,n}^p a_n^p \end{aligned}\right\} \quad \text{for} \quad p \geqq 1,$$

and

$$\left.\begin{aligned} &(3') \qquad \sum_{n=1}^{\infty} \lambda_n^{1-p} A_{n,\infty}^p a_n^p \leqq 8 \sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \\ &(4') \qquad \sum_{n=1}^{\infty} \lambda_n^{1-p} A_{1,n}^p a_n^p \leqq 9 \sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \end{aligned}\right\} \quad \textit{for} \quad 0 < p \leqq 1.$$

Inequalities (1′)—(4′) reduce to those of HARDY and LITTLEWOOD if $\lambda_n = n^{-c}$, with $c > 1$ in (1′) and (2′), and with $c \leqq 1$ in (3′) and (4′).

Choosing $\lambda_n = n^{-1}$ or $\lambda_n = n^{-1}(\log(n+1))^{-1-\varepsilon}$ ($\varepsilon > 0$) we obtain as further special cases the inequalities:

$$\left.\begin{aligned} &(1'') \qquad \sum_{n=1}^{\infty} \frac{1}{n} A_{n,\infty}^p \leqq K \sum_{n=1}^{\infty} \frac{(\log(n+1))^p}{n} (na_n)^p \\ &(2'') \qquad \sum_{n=1}^{\infty} \frac{1}{n(\log(n+1))^{1+\varepsilon}} A_{1,n}^p \leqq K \sum_{n=1}^{\infty} \frac{(\log(n+1))^p}{n(\log(n+1))^{1+\varepsilon}} (na_n)^p \end{aligned}\right\} \quad \textit{for} \quad p \geqq 1$$

and

$$\left.\begin{aligned} &(3'') \qquad \sum_{n=1}^{\infty} \frac{(\log n)^p}{n} (na_n)^p \leqq K \sum_{n=1}^{\infty} \frac{1}{n} A_{n,\infty}^p \\ &(4'') \qquad \sum_{n=1}^{\infty} \frac{(\log n)^p}{n(\log(n+1))^{1+\varepsilon}} (na_n)^p \leqq K \sum_{n=1}^{\infty} \frac{A_{1,n}^p}{n(\log(n+1))^{1+\varepsilon}} \end{aligned}\right\} \quad \textit{for} \quad 0 < p \leqq 1.$$

Let $L(x)$ $(x > 0)$ be a "slowly increasing" or "slowly decreasing" function, that is, let $L(x)$ be a positive increasing or decreasing function with the property:

$$\lim_{x\to\infty} \frac{L(kx)}{L(x)} = 1 \quad \text{for every} \quad k > 0.$$

A standard computation gives that for such a function $L(x)$ we have

$$\sum_{k=n}^{\infty} \frac{L(k)}{k^{1+\varepsilon}} \leqq K \frac{nL(n)}{n^{1+\varepsilon}}$$

and

$$\sum_{k=1}^{n} \frac{L(k)}{k^{1-\varepsilon}} \leqq K \frac{nL(n)}{n^{1-\varepsilon}}$$

for any $\varepsilon > 0$.

Using these estimates, for $\lambda_n = n^{-1\pm\varepsilon} L(n)$ inequalities (1′)—(4′) take on the form:

$$\left.\begin{aligned} &(1''') \qquad \sum_{n=1}^{\infty} \frac{L(n)}{n^{1+\varepsilon}} A_{1,n}^p \leqq K \sum_{n=1}^{\infty} \frac{L(n)}{n^{1+\varepsilon}} (na_n)^p \\ &(2''') \qquad \sum_{n=1}^{\infty} \frac{L(n)}{n^{1-\varepsilon}} A_{n,\infty}^p \leqq K \sum_{n=1}^{\infty} \frac{L(n)}{n^{1-\varepsilon}} (na_n)^p \end{aligned}\right\} \quad \textit{for} \quad p \geqq 1,$$

and

$$\left.\begin{aligned}&(3''')\qquad \sum_{n=1}^{\infty}\frac{L(n)}{n^{1+\varepsilon}}(na_n)^p \leqq K\sum_{n=1}^{\infty}\frac{L(n)}{n^{1+\varepsilon}}A_{1,n}^p\\ &(4''')\qquad \sum_{n=1}^{\infty}\frac{L(n)}{n^{1-\varepsilon}}(na_n)^p \leqq K\sum_{n=1}^{\infty}\frac{L(n)}{n^{1-\varepsilon}}A_{n,\infty}^p\end{aligned}\right\}\quad \textit{for}\quad 0<p\leqq 1.$$

Proofs

Inequality (1). We may assume that the second member has finite value and that not all a_n vanish. For $p=1$ the situation is obvious: we have only to interchange the order of the summations. For $p>1$ we follow analogous lines as in [3], p. 210. We use the notations:

$$\alpha_0=0,\quad \alpha_n=A_{\nu_{n-1}+1,\nu_n},\quad \beta_n=\sum_{k=1}^{n}\alpha_k,\quad \varrho_n=\lambda_{\nu_n},$$

$$R_n=\Lambda_{\nu_n,\infty}\quad\text{and}\quad d_n=R_n\varrho_n^{-1}$$

for every $1\leqq n\leqq N$. If $N<\infty$ then let

$$\varrho_{N+1}=d_{N+1}=0.$$

Then, for any positive integer $m\,(\leqq N)$, we have

$$\sum_{k=1}^{\nu_m}\lambda_k A_{1,k}^p=\sum_{n=1}^{m}\varrho_n\beta_n^p=\sum_{n=1}^{m}(d_n\varrho_n-d_{n+1}\varrho_{n+1})\beta_n^p\leqq\sum_{n=1}^{m}\varrho_n d_n(\beta_n^p-\beta_{n-1}^p)\leqq$$

$$\leqq p\sum_{n=1}^{m}\varrho_n d_n\alpha_n\beta_n^{p-1}\leqq p\left\{\sum_{n=1}^{m}\varrho_n(d_n\alpha_n)^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{m}\varrho_n\beta_n^p\right\}^{\frac{p-1}{p}}.$$

Hence

$$\left\{\sum_{k=1}^{\nu_m}\lambda_k A_{1,k}^p\right\}^{\frac{1}{p}}\leqq p\left\{\sum_{n=1}^{m}\varrho_n^{1-p}R_n^p\alpha_n^p\right\}^{\frac{1}{p}}=p\left\{\sum_{n=1}^{m}\lambda_{\nu_n}^{1-p}\Lambda_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p\right\}^{\frac{1}{p}},$$

which gives (1).

Inequality (2). For $p=1$ the assertion is obvious. For $p>1$ we proceed as follows. We suppose that the second member has a finite value (which is the only case of interest). For $1\leqq n\leqq N$ we set

$$\alpha_n=A_{\nu_n,\nu_{n+1}-1},\quad \beta_n=\sum_{k=n}^{N}\alpha_k,\quad \varrho_n=\lambda_{\nu_n},$$

$$R_n=\Lambda_{1,\nu_n}=\sum_{k=1}^{n}\varrho_k.$$

If $N<\infty$ then we have

$$\sum_{n=1}^{\infty}\lambda_n A_{n,\infty}^p=\sum_{n=1}^{N}\varrho_n\beta_n^p=\sum_{n=1}^{N-1}R_n(\beta_n^p-\beta_{n+1}^p)+R_N\beta_N^p\leqq$$
$$\leqq\sum_{n=1}^{N}R_np\alpha_n\beta_n^{p-1}\leqq p\left\{\sum_{n=1}^{N}\varrho_n^{1-p}R_n^p\alpha_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{N}\varrho_n\beta_n^p\right\}^{\frac{p-1}{p}},$$

which gives (2).

If $N=\infty$ then we show first that

$$A_{n,\infty}\to 0\quad\text{as}\quad n\to\infty. \tag{5}$$

To prove (5) we have only to combine the estimate

$$A_{\nu_n,\infty}=\sum_{k=n}^{\infty}\varrho_k^{\frac{1}{p}-1}R_kR_k^{-1}\varrho_k^{1-\frac{1}{p}}\alpha_k\leqq\left\{\sum_{k=n}^{\infty}\varrho_k^{1-p}R_k^p\alpha_k^p\right\}^{\frac{1}{p}}\left\{\sum_{k=n}^{\infty}\varrho_kR_k^{\frac{p}{1-p}}\right\}^{\frac{p-1}{p}},$$

obtained by Hölder's inequality, with the fact that the sum

$$\sum_{k=1}^{\infty}\varrho_kR_k^{\frac{p}{1-p}}$$

is of finite value (apply the theorem of Abel, cf. [2], Theorem 162).

Now fix two natural numbers, m and M, and choose k such that

$$\beta_{k-1}\leqq\beta_mM^{-\frac{1}{p}};$$

this is seen to be possible by (5). Then we have

$$\sum_{n=1}^{\nu_k}\lambda_nA_{n,\infty}^p=\sum_{n=1}^{k}\varrho_n\beta_n^p=\sum_{n=1}^{k-1}R_n(\beta_n^p-\beta_{n+1}^p)+R_k\beta_k^p\leqq$$
$$\leqq\sum_{n=1}^{k-1}R_n(\beta_n^p-\beta_{n+1}^p)+\left(\sum_{n=1}^{m}\varrho_n\right)\beta_m^pM^{-1}+\left(\sum_{n=m+1}^{k}\varrho_n\right)\beta_k^p\leqq$$
$$\leqq\sum_{n=1}^{k-1}R_np\alpha_n\beta_n^{p-1}+M^{-1}\sum_{n=1}^{m}\varrho_n\beta_n^p+\sum_{n=m+1}^{k}\varrho_n\beta_n^p.$$

Hence

$$(1-M^{-1})\sum_{n=1}^{m}\varrho_n\beta_n^p\leqq\sum_{n=1}^{k-1}R_np\alpha_n\beta_n^{p-1}\leqq p\left\{\sum_{n=1}^{\infty}\varrho_n^{1-p}R_n^p\alpha_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\varrho_n\beta_n^p\right\}^{\frac{p-1}{p}}.$$

Since m and M may be arbitrarily large this estimate gives

$$\sum_{n=1}^{\infty}\varrho_n\beta_n^p\leqq p\left\{\sum_{n=1}^{\infty}\varrho_n^{1-p}R_n^p\alpha_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\varrho_n\beta_n^p\right\}^{\frac{p-1}{p}},$$

and hence we get (2) immediately.

The constant p^p in (1) and (2) is the best possible because it is so even if $\lambda_n = n^{-p}$ in (1) and if $\lambda_n = 1$ in (2); see [2], Theorem 326 and Theorem 331.

Inequality (3). Again we may assume that the second member is finite, and that not all a_n and λ_n vanish. The case $p=1$ is trivial. In case $0<p<1$ let us remark first that our assumption imply that $\Lambda_{1,\infty}<\infty$ and hence $\Lambda_{n,\infty}\to 0$ as $n\to\infty$. Now we define an increasing sequence $m_0, m_1, \ldots$ of integers as follows. Let $m_0=0$. If m_n is already defined, we choose for m_{n+1} the smallest natural number $k>m_n$ for which $\Lambda_{m_n+1,k}\geqq \Lambda_{k+1\infty}$. This is possible because $\Lambda_{n,\infty}\to 0$. So we will have

$$\Lambda_{m_n+1,m_{n+1}} \geqq \Lambda_{m_{n+1}+1,\infty} \tag{6}$$

and

$$\Lambda_{m_n+1,m_{n+1}} < 2\lambda_{m_{n+1}} + \Lambda_{m_{n+1}+1,m_{n+2}} + \Lambda_{m_{n+2}+1,\infty} \leqq 2\lambda_{m_{n+1}} + 2\Lambda_{m_{n+1}+1,m_{n+2}},$$

and hence

$$\Lambda_{m_n+1,m_{n+1}} \leqq 2\Lambda_{m_{n+1},m_{n+2}}. \tag{7}$$

Set

$$\alpha_n = A_{\nu_{n-1}+1,\nu_n}, \quad \varrho_n = \lambda_{\nu_n} \quad \text{and} \quad R_n = \Lambda_{n,\infty}.$$

Using (6) and (7) we obtain

$$\sum_{m_n<\nu_k\leqq m_{n+1}}^{(k)} \varrho_k^{1-p} R_{\nu_k}^p \alpha_k^p \leqq \sum_{m_n<\nu_k\leqq m_{n+1}}^{(k)} \varrho_k^{1-p} R_{m_n+1}^p \alpha_k^p \leqq {}^{*)}$$

$$\leqq R_{m_n+1}^p \Big\{ \sum_{m_n<\nu_k\leqq m_{n+1}}^{(k)} \alpha_k \Big\}^p \Big\{ \sum_{m_n<\nu_k\leqq m_{n+1}}^{(k)} \varrho_k \Big\}^{1-p} \leqq R_{m_n+1} \Big\{ \sum_{m_n<\nu_k\leqq m_{n+1}}^{(k)} \alpha_k \Big\}^p \leqq$$

$$\leqq 2\left(\sum_{i=m_n+1}^{m_{n+1}} \lambda_i \right) \Big\{ \sum_{m_n<\nu_k\leqq m_{n+1}}^{(k)} \alpha_k \Big\}^p \leqq 4 \left(\sum_{i=m_n+1}^{m_{n+2}} \lambda_i \right) \Big\{ \sum_{m_n<\nu_k\leqq m_{n+1}}^{(k)} \alpha_k \Big\}^p \leqq 4 \sum_{i=m_n+1}^{m_{n+2}} \lambda_i A_{1,i}^p.$$

From this we get

$$\sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p \leqq 4 \sum_{n=0}^{\infty} \sum_{i=m_n+1}^{m_{n+2}} \lambda_i A_{1,i}^p \leqq 8 \sum_{k=1}^{\infty} \lambda_k A_{1,k}^p$$

which is the required inequality (3).

Inequality (4). Here we distinguish two cases according as $\Sigma\lambda_n$ has finite or infinite value.

If $\Sigma\lambda_n<\infty$ we define a sequence of integers $\mu_0, \mu_1, \ldots$. We set $\mu_0=0$, $\mu_1=1$, and if μ_n has already been defined we choose $\mu'_{n+1}=k$, where $k\ (>\mu_n)$ denotes the smallest integer satisfying

$$\Lambda_{\mu_n+1,k} \geqq 3\Lambda_{\mu_{n-1}+1,\mu_n} \tag{8}$$

provided such a k exists. If $\mu'_{n+1}>\mu_n+1$ then let $\mu_{n+1}=\mu'_{n+1}-1$, and if $\mu'_{n+1}=\mu_n+1$ then let $\mu_{n+1}=\mu_n+1$. If there exists no natural number k with (8) then let $\mu_{n+1}=\infty$.

*) $\sum\limits_{m_n<\nu_k\leqq m_{n+1}}^{(k)}$ denotes a sum for all k for which $m_n<\nu_k\leqq m_{n+1}$ is satisfied.

It is clear that this inductive definition always stops at some $n=N_0$, that is, $\mu_{N_0}=\infty$ holds. For in the opposite case, by the definition of μ_n, the inequality

$$3I_{n-2} \leqq I_{n-1}+I_n \tag{9}$$

holds for all $2\leqq n<N_0$, where $I_n=\Lambda_{\mu_n+1,\mu_{n+1}}$, and inequality (9) for infinitely many n would imply $\Sigma\lambda_k=\infty$, contrary to the assumption. By (9) we have for $1\leqq n<N_0-1$

$$\sum_{i=1}^{\mu_n}\lambda_i \leqq 3I_{n-1}+I_n. \tag{10}$$

Set

$$\alpha_n = A_{\nu_n,\nu_{n+1}-1}, \quad \beta_n = \sum_{k=n}^{\infty} a_k \quad \text{and} \quad R_n = \sum_{i=1}^{n}\lambda_i. \tag{11}$$

By the definition of the sequence $\{\mu_n\}$ and by (10), for $2\leqq n<N_0-1$ we have

$$\sum_{\mu_n<\nu_k\leqq\mu_{n+1}}^{(k)} \lambda_{\nu_k}^{1-p}R_{\nu_k}^p\alpha_k^p \leqq R_{\mu_{n+1}}^p\Big(\sum_{\mu_n<\nu_k\leqq\mu_{n+1}}^{(k)}\lambda_{\nu_k}\Big)^{1-p}\Big(\sum_{\mu_n<\nu_k\leqq\mu_{n+1}}^{(k)}\alpha_k\Big)^p \leqq$$

$$\leqq\left(R_{\mu_{n-1}}+\sum_{i=\mu_{n-1}+1}^{\mu_{n+1}}\lambda_i\right)\Big(\sum_{\mu_n<\nu_k\leqq\mu_{n+1}}^{(k)}\alpha_k\Big)^p\leqq$$

$$\leqq\left(3\sum_{i+\mu_{n-2}+1}^{\mu_{n-1}}\lambda_i+2\sum_{i=\mu_{n-1}+1}^{\mu_n}\lambda_i+\sum_{i=\mu_n+1}^{\mu_{n+1}}\lambda_i\right)\Big(\sum_{\mu_n<\nu_k\leqq\mu_{n+1}}^{(k)}\alpha_k\Big)^p\leqq$$

$$\leqq 3\sum_{i=\mu_{n-2}+1}^{\mu_{n-1}}\lambda_i\beta_i^p+5\sum_{i=\mu_{n-1}+1}^{\mu_n}\lambda_i\beta_i^p+\lambda_{\mu_n+1}\beta_{\mu_{n+1}}^p\leqq$$

$$\leqq 3\sum_{i=\mu_{n-2}+1}^{\mu_{n-1}}\lambda_i\beta_i^p+5\sum_{i=\mu_{n-1}+1}^{\mu_n}\lambda_i\beta_i^p+\sum_{i=\mu_n+1}^{\mu_{n+1}}\lambda_i\beta_i^p.$$

An easy computation gives by (8) and (10) that

$$\sum_{\mu_1\leqq\nu_k\leqq\mu_2}^{(k)}\lambda_{\nu_k}^{1-p}R_{\nu_k}^p\alpha_k^p\leqq 5\lambda_1A_{1,\infty}^p+\lambda_2A_{2,\infty}^p,$$

and that

$$\sum_{\mu_{N_0-1}<\nu_k<\infty}^{(k)}\lambda_{\nu_k}^{1-p}R_{\nu_k}^p\alpha_k^p\leqq 3\sum_{i=\mu_{N_0-3}+1}^{\mu_{N_0-2}}\lambda_i\beta_i^p+5\sum_{i=\mu_{N_0-2}+1}^{\mu_{N_0-1}}\lambda_i\beta_i^p.$$

Summing up our results we get (4) if $\sum_{i=1}^{\infty}\lambda_i<\infty$.

If $\sum_{i=1}^{\infty}\lambda_i=\infty$ then we define an increasing sequence $\{m_n\}$ of integers as follows. Let $m_0=0$ and $m_1=1$. If $m_0, m_1, \ldots, m_n$ $(n\geqq 1)$ have been defined, then let m_{n+1} be the smallest integer $k>m_n$ for which

$$\Lambda_{m_n+1,k}\geqq 2\Lambda_{m_{n-1}+1,m_n}.$$

So we have

$$I_{n+1} \geqq 2I_n, \quad \text{where} \quad I_n = \Lambda_{m_{n-1}+1, m_n}. \tag{12}$$

Using the notations of (11), we have by (12) for every $n(\geqq 1)$

$$\sum_{m_n \leqq \nu_k < m_{n+1}}^{(k)} \lambda_{\nu_k}^{1-p} R_{\nu_k}^p \alpha_k^p \leqq R_{m_{n+1}-1} \Big(\sum_{m_n \leqq \nu_k < m_{n+1}}^{(k)} \alpha_k \Big)^p \leqq 4 \sum_{i=m_{n-1}+1}^{m_n} \lambda_i \beta_i^p.$$

Hence we obtain that

$$\sum_{k=1}^{\mu} \lambda_{\nu_k}^{1-p} \Lambda_{1,\nu_k}^p A_{\nu_k,\nu_{k+1}-1}^p = \sum_{n=1}^{\infty} \sum_{m_n \leqq \nu_k < m_{n+1}}^{(k)} \lambda_{\nu_k}^{1-p} \Lambda_{1,\nu_k}^p A_{\nu_k,\nu_{k+1}-1}^p \leqq$$

$$\leqq 4 \sum_{n=1}^{\infty} \sum_{i=m_{n-1}+1}^{m_n} \lambda_i A_{i,\infty}^p \leqq 4 \sum_{k=1}^{\infty} \lambda_k A_{k,\infty}^p,$$

which gives (4).

The proof is complete.

References

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